Predicate logic

We shall now present a more general logic with more expressive power than propositional logic.

E.g., the following argument, though valid, cannot be formalised in propositional logic:

All primates are mammals.
Humans are primates.
Therefore, humans are mammals.

Predicate or first order logic, on the other hand, has enough expressive power to formalise all of mathematics.

Definition A first order language L consists of

(i) Logical symbols :

logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
variables $x_0, x_1, \ldots$
quantifiers $\exists$ and $\forall$
equality symbol $=$

(ii) Extra logical symbols:

a set of constant symbols $\{c_i \mid i \in I\}$

for every n, a set of n-ary function symbols $\{f_j \mid j \in J_n\}$

for every n, a set of n-ary relation symbols $\{R_\ell \mid \ell \in H_n\}$.

<u>Terms of a language:</u>

Given a logical language L as above, the set of <u>L-terms</u> is the smallest set st.

- every constant symbol c is an L-term
- every variable x is an L-term
- if $t_1, \dots, t_n$ are L-terms and f is an n-ary function symbol, then

$$f(t_1, \dots, t_n)$$

is an L-term.

__Example__ If L is a language with constant symbols $0, 1$ and binary function symbols f and g, then

$$x_0 \quad , \quad f(x_0, 0) \;, \quad g(1, f(x_0, 0))$$

are all terms of the language L.

__Convention__ Sometimes, when using wellknown function symbols such as $+, -, \cdot$, we allow ourselves to write

$$t_1 + t_2$$

rather than $+(t_1, t_2)$.

So $x_0 + x_1$, $(x_7 + 0) - 1$ are all terms of the language L having constant symbols $0, 1$ and function symbols $+, -$.

Substitution in terms:

Suppose L is a first order language, $v_1, \dots, v_n$ are distinct variables and $t_1, \dots, t_n$ are L-terms. We define the substitution of terms $t_1, \dots, t_n$ for variables $v_1, \dots, v_n$ in a term s by induction on the construction of s:

- If s is a constant, set $s[t_1/v_1, \dots, t_n/v_n] = s$.

- If s is a variable $v \neq v_1, \dots, v_n$, set
  $$s[t_1/v_1, \dots, t_n/v_n] = s .$$

- If $s = v_i$, set $s[t_1/v_1, \dots, t_n/v_n] = t_i$.

- If $s = f(s_1, \dots, s_m)$, where f is an m-ary function and $s_1, \dots, s_m$ are terms, then
  $$s[t_1/v_1, \dots, t_n/v_n] = f(s_1[t_1/v_1, \dots, t_n/v_n], \dots, s_m[t_1/v_1, \dots, t_n/v_n]).$$

We say that $s[t_1/v_1, \ldots, t_n/v_n]$ is obtained from s by <u>simultaneous</u> <u>substitution</u> of $t_1, \ldots, t_n$ for $v_1, \ldots, v_n$.

<u>Exercise</u> Give an example to show that in general

$$s[t_1/v_1, t_2/v_2] \neq (s[t_1/v_1])[t_2/v_2].$$

In other words, successive substitution is <u>not</u> the same as simultaneous substitution.

<u>Lemma</u> If $s, t_1, \ldots, t_n$ are L-terms and $v_1, \ldots, v_n$ are distinct variables, then

$$s[t_1/v_1, \ldots, t_n/v_n]$$

is an L-term.

<u>Proof</u> This is proved by induction on the construction of s. $\square$